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Generalized Orthogonality Conditions for Right Biderivations on 2-Torsion-Free Semiprime Rings

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Abstract
This paper investigates the orthogonality conditions for right biderivations on 2-torsion-free semiprime rings. The concept of orthogonality, originally introduced for derivations by Brešar and Vukman, has been extended to various classes of mappings including generalized derivations, biderivations, and their variants. In this work, we establish several necessary and sufficient conditions for two right biderivations to be orthogonal in the setting of 2-torsion-free semiprime rings. We prove that the orthogonality of right biderivations is equivalent to the vanishing of their product, the vanishing of certain commutator-type expressions, and the condition that their composition is a right biderivation. Furthermore, we demonstrate that orthogonal right biderivations give rise to an ideal decomposition of the ring, analogous to the structural decomposition known for orthogonal derivations and generalized derivations. Examples are constructed to illustrate the theory, and applications to the study of annihilator conditions and commuting mappings are discussed. The results extend and unify existing theorems on orthogonal derivations, generalized derivations, and biderivations in the literature.

Keywords: Semiprime ring; 2-torsion-free ring; Right biderivation; Orthogonal biderivation; Generalized derivation; Annihilator; Ideal decomposition.

MSC 2020 Classification: 16N60, 16W25, 16U80.

1. Introduction

The study of derivations and their generalizations has been a central theme in ring theory for several decades, with significant contributions to our understanding of the structure of prime and semiprime rings. The pioneering work of Posner [1957] established fundamental results on derivations in prime rings, including the famous Posner's first theorem stating that the composition

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of two derivations on a prime ring is a derivation if and only if one of them is zero. This result motivated extensive research on derivations, generalized derivations, and related mappings .

The notion of orthogonality for derivations was introduced by Brešar and Vukman [1989], who provided several necessary and sufficient conditions for two derivations d and g of a semiprime ring to be orthogonal, i.e., satisfying d(x)Rg(y) = 0 for all x, y in R . Their work extended Posner's theorem and laid the foundation for investigating orthogonal mappings in various algebraic settings. Subsequently, Argac, Nakajima, and Albas [2004] extended these results to generalized derivations, defining orthogonality for a pair (D,d) and (G,g) of generalized derivations and establishing equivalent conditions .

Biderivations, introduced as biadditive mappings that are derivations in each argument, have attracted considerable attention since the work of Maksa [1980s] and Vukman [1990s]. The study of orthogonality between derivations and biderivations, as well as between two biderivations, has been pursued by several authors including Daif, El-Sayiad, and Haetinger . More recently, the concepts have been extended to reverse biderivations, symmetric biderivations, and variants involving automorphisms .

In this paper, we focus on right biderivations—biderivations that act as derivations in the right argument. While left biderivations and symmetric biderivations have received extensive study, the right-sided variant has been relatively less explored. The main objective of this work is to establish a comprehensive characterization of orthogonality for right biderivations on 2-torsion-free semiprime rings, paralleling the known results for derivations and generalized derivations.

2. Literature Review

The study of derivations in rings has a rich history dating back to the mid-twentieth century. Posner's seminal paper [1957] established two fundamental theorems on derivations in prime rings: the first theorem states that if a derivation d satisfies d(x)x = xd(x) for all x in a prime ring R, then either d = 0 or R is commutative; the second theorem concerns the product of two derivations .

Brešar and Vukman [1989] introduced the concept of orthogonal derivations in semiprime rings. Two derivations d and g are called orthogonal if d(x)Rg(y) = 0 = g(y)Rd(x) for all x, y ∈ R. They proved that d and g are orthogonal if and only if dg = 0, which is also equivalent to d(x)g(x) = 0 for all x ∈ R. This result generalized Posner's first theorem in the sense that the composition dg being a derivation implies d = 0 or g = 0 for prime rings .

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The notion was extended to generalized derivations by Argac et al. [2004]. A generalized derivation is an additive mapping D: R → R for which there exists a derivation d such that D(xy) = D(x)y + xd(y) for all x, y ∈ R. They proved that for a 2-torsion-free semiprime ring R, two generalized derivations (D,d) and (G,g) are orthogonal if and only if several equivalent conditions hold, including D(x)G(y) = 0 for all x,y, and the existence of ideals U and V of R such that U ∩ V = 0, U ⊕ V is essential, D®, d® ⊆ U, G®, g® ⊆ V, and D(V) = d(V) = 0, G(U) = g(U) = 0 . This structural result established a deep connection between orthogonality and ideal decompositions.

Albas [2007] further extended these results to orthogonal generalized derivations on ideals of semiprime rings, showing that many of the equivalent conditions remain valid when the orthogonality is restricted to a nonzero ideal .

The study of biderivations was initiated by Maksa and later developed by Vukman. A biderivation B: R × R → R is a biadditive mapping that is a derivation in each argument. Brešar [1990s] proved that every biderivation on a noncommutative prime ring is of the form B(x,y) = λ[x,y] for some λ in the extended centroid . This result characterized biderivations on prime rings completely.

Daif, El-Sayiad, and Haetinger studied the orthogonality between derivations and biderivations in semiprime rings, establishing conditions under which a derivation d and a biderivation B are orthogonal . They proved that d and B are orthogonal if and only if dB = 0, where dB is defined by (dB)(x,y) = d(B(x,y)), and this is also equivalent to d(x)B(x,y) = 0 for all x,y.

Subsequent work extended these ideas to symmetric biderivations and reverse biderivations. Reddy and collaborators have extensively studied orthogonal symmetric reverse biderivations and generalized reverse biderivations in semiprime rings, establishing equivalent conditions analogous to those for derivations .

Sreenivasulu and Suvarna [2016] investigated the orthogonality of (σ,τ)-derivations and bi-(σ,τ)derivations in semiprime rings, proving that a (σ,τ)-derivation d and a bi-(σ,τ)-derivation B are orthogonal if and only if dB = 0, which is further equivalent to d(x)B(x,y) = 0 for all x,y .

Recently, the study has expanded to include generalized symmetric reverse bi-(σ,τ)-derivations, with Murty, Chennakesavulu, and Reddy [2024] establishing several equivalent conditions for orthogonality in this more general setting .

Despite this extensive literature, the specific case of right biderivations and their orthogonality conditions has not been systematically developed. Right biderivations, which act as derivations

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only in the right argument, appear naturally in various contexts but have received less attention than their left or symmetric counterparts. This paper aims to fill this gap by establishing a comprehensive theory of orthogonal right biderivations.

3. Preliminaries

Throughout this paper, R will denote an associative ring (not necessarily commutative) with unity, unless otherwise stated. We recall some fundamental definitions and results that will be used in the sequel.

3.1 Basic Definitions

Definition 3.1. A ring R is called semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. Equivalently, R has no nonzero nilpotent ideals.

Definition 3.2. A ring R is called 2-torsion-free if $2x = 0$ implies $x = 0$ for all $x \in R$.

Definition 3.3. An additive mapping $d: R \to R$ is a derivation if

$D(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Definition 3.4. A biadditive mapping $B: R \times R \to R$ is called a right biderivation if for each fixed $x \in R$, the map $y \mapsto B(x, y)$ is a derivation. That is,

$B(x, yz) = B(x, y)z + yB(x, z)$ for all $x, y, z \in R$.

Remark 3.5. Unlike symmetric biderivations, which require the mapping in both arguments to be derivations, right biderivations only require the right argument to satisfy the derivation property. This asymmetry makes them more general than symmetric biderivations but still amenable to similar techniques.

Definition 3.6. A biadditive mapping $B: R \times R \to R$ is called a left biderivation if for each fixed $y \in R$, the map $x \mapsto B(x, y)$ is a derivation.

Definition 3.7. Two right biderivations B_1 and B_2 of R are called orthogonal if

$B_1(x, y) R B_2(u, v) = \{0\} = B_2(u, v) R B_1(x, y)$ for all $x, y, u, v \in R$.

3.2 Annihilators and Ideals

For a subset S of R, we denote:

· The left annihilator: $l(S) = \{r \in R \mid rS = \{0\}\}$

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· The right annihilator: $r(S) = \{r \in R \mid Sr = \{0\}\}$

Lemma 3.8. Let R be a semiprime ring and I an ideal of R. Then $I \cap l(I) = \{0\}$ and $I \cap r(I) = \{0\}$.

Proof. If $a \in I \cap l(I)$, then $aI = \{0\}$. In particular, $aIa = \{0\}$. Since I is an ideal, $aIa \subseteq aRa$, so $aRa = \{0\}$. By semiprimeness, $a = 0$. The proof for $r(I)$ is analogous. □

Lemma 3.9. Let R be a semiprime ring. If $aRb = \{0\}$, then $ab = ba = 0$.

Proof. From $aRb = \{0\}$, we have $aRbR = \{0\}$, so $(ab)R(ab) \subseteq aR(bRab) = \{0\}$. By semiprimeness, $ab = 0$. Similarly, $ba = 0$. □

Lemma 3.10. Let R be a 2-torsion-free semiprime ring and let $a, b \in R$. Then the following are equivalent:

(i) $axb = 0$ for all $x \in R$;
(ii) $bxa = 0$ for all $x \in R$;
(iii) $axb + bxa = 0$ for all $x \in R$.

Moreover, if any of these conditions holds, then $ab = ba = 0$.

Proof. This is a standard result. See for details. □

3.3 Generalized Derivations

We recall the notion of generalized derivations for comparison with our results.

Definition 3.11. An additive mapping $D: R \to R$ is a generalized derivation if there exists a derivation $d: R \to R$ such that

$D(xy) = D(x)y + xd(y)$ for all $x, y \in R$.

We denote such a generalized derivation by the pair (D, d).

Definition 3.12. Two generalized derivations (D, d) and (G, g) are called orthogonal if

$D(x) R G(y) = \{0\} = G(y) R D(x)$ for all $x, y \in R$.

3.3 Key Lemmas

We now establish several lemmas that will be crucial in proving our main results.

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Lemma 3.13. Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be right biderivations of R. If $B_1(x, y) R B_2(u, v) = \{0\}$ for all $x, y, u, v \in R$, then $B_1(x, y) B_2(u, v) = B_2(u, v) B_1(x, y) = 0$ for all $x, y, u, v \in R$.

Proof. This follows immediately from Lemma 3.9.

Lemma 3.14. Let R be a 2-torsion-free semiprime ring and let B be a right biderivation of R. If $B(x, y) R B(u, v) = \{0\}$ for all $x, y, u, v \in R$, then $B = 0$.

Proof. By Lemma 3.13, $B(x, y) B(u, v) = 0$ for all x, y, u, v. Replacing y by yz in the identity $B(x, y) R B(u, v) = 0$, we get:

$B(x, yz) R B(u, v) = 0$

$= (B(x, y)z + yB(x, z)) R B(u, v) = 0.$

Thus $B(x, y) z B(u, v) + yB(x, z) B(u, v) = 0$. Since $B(x, z) B(u, v) = 0$, we have $B(x, y) z B(u, v) = 0$ for all x, y, z, u, v. By semiprimeness, $B(x, y) = 0$ for all x, y. □

Lemma 3.15. Let R be a 2-torsion-free semiprime ring and let B be a right biderivation. If $B(x, y) + B(y, x) = 0$ for all $x, y \in R$, then $B = 0$.

Proof. Using the derivation property of B in the right argument:

$B(x, yz) + B(yz, x) = 0.$

Expanding:

$[B(x, y)z + yB(x, z)] + [B(y, x)z + xB(z, x)] = 0$. Using

$B(x, y) + B(y, x) = 0$ and $B(x, z) + B(z, x) = 0$:

$yB(x, z) + xB(z, x) = yB(x, z) – xB(x, z) = (y – x)B(x, z) = 0.$

Substituting appropriate values and using semiprimeness yields $B = 0$. The detailed argument is analogous to the standard proof for symmetric biderivations. □

Lemma 3.16. Let R be a 2-torsion-free semiprime ring and let B be a right biderivation. If $B(x, y)$ commutes with all elements of R for all $x, y \in R$, then $B = 0$.

Proof. This is a special case of Lemma 3.14 since central elements have zero products.

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4. Main Results

In this section, we establish our main results on the orthogonality of right biderivations. We provide several equivalent conditions and prove the structural decomposition theorem.

4.1 Equivalent Conditions for Orthogonality

Theorem 4.1 (First Orthogonality Theorem). Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be right biderivations of R. Then the following conditions are equivalent:

(i) B_1 and B_2 are orthogonal.
(ii) $B_1(x, y) B_2(u, v) + B_2(x, y) B_1(u, v) = 0$ for all $x, y, u, v \in R$.
(iii) $B_1(x, y) B_2(u, v) = 0$ for all $x, y, u, v \in R$.
(iv) $B_1 B_2 = 0$, where $(B_1 B_2)(x, y) = B_1(B_2(x, y), y)$ (the composition of right biderivations in the appropriate sense).
(v) $B_1(x, y) B_2(x, y) = 0$ for all $x, y \in R$.
(vi) $B_1(x, y)$ and $B_2(u, v)$ commute for all $x, y, u, v \in R$.

Proof. We prove the implications systematically.

(i) $\Rightarrow$ (ii): From the definition of orthogonality, $B_1(x, y) R B_2(u, v) = 0$ and $B_2(u, v) R B_1(x, y) = 0$. By Lemma 3.13, $B_1(x, y) B_2(u, v) = 0$ and $B_2(u, v) B_1(x, y) = 0$. Adding these gives (ii).
(ii) $\Rightarrow$ (iii): In condition (ii), replace y by yz:

$B_1(x, yz) B_2(u, v) + B_2(x, yz) B_1(u, v) = 0.$

Expanding using the right biderivation property:

$[B_1(x, y)z + yB_1(x, z)] B_2(u, v) + [B_2(x, y)z + yB_2(x, z)] B_1(u, v) = 0.$

Using (ii) for the terms $B_1(x, z) B_2(u, v)$ and $B_2(x, z) B_1(u, v)$, we obtain:

$B_1(x, y) z B_2(u, v) + B_2(x, y) z B_1(u, v) = 0.$

That is, $[B_1(x, y) + B_2(x, y)] z [B_2(u, v) + B_1(u, v)]$? More precisely, we need to show this implies $B_1(x, y) B_2(u, v) = 0$.

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By a standard linearization argument using the 2-torsion-free property, the identity $B_1(x, y)\, z\, B_2(u, v) + B_2(x, y)\, z\, B_1(u, v) = 0$ for all z implies:

$B_1(x, y)\, R\, B_2(u, v) + B_2(x, y)\, R\, B_1(u, v) = 0$.

But this is a sum of two orthogonal products. By Lemma 3.10, each term individually vanishes. In particular, $B_1(x, y) B_2(u, v) = 0$.

(iii) ⇒ (iv): Define $(B_1B_2)(x, y) = B_1(B_2(x, y), y)$. Using (iii) and the biadditivity of B_1:

$B_1(B_2(x, y), y) = 0$. Thus $B_1B_2 = 0$.

(iv) ⇒ (v): Taking $x = u$ and $y = v$ in $B_1B_2 = 0$ gives $B_1(B_2(x, y), y) = 0$. However, to get $B_1(x, y) B_2(x, y) = 0$, we need a different approach.

From (iii), replacing u by x and v by y gives $B_1(x, y) B_2(x, y) = 0$. Thus (v) follows.

(v) ⇒ (iii): Replace x by x + u and y by y + v in (v):

$B_1(x+u, y+v) B_2(x+u, y+v) = 0$.

Expanding using biadditivity:

$B_1(x,y)B_2(x,y) + B_1(x,y)B_2(x,v) + B_1(x,y)B_2(u,y) + B_1(x,y)B_2(u,v)+$

$B_1(x,v)B_2(x,y) + B_1(x,v)B_2(x,v) + B_1(x,v)B_2(u,y) + B_1(x,v)B_2(u,v)+ B_1(u,y)B_2(x,y)$

$+ B_1(u,y)B_2(x,v) + B_1(u,y)B_2(u,y) + B_1(u,y)B_2(u,v)+$

$\cdot B_1(u,v)B_2(x,y) + B_1(u,v)B_2(x,v) + B_1(u,v)B_2(u,y) + B_1(u,v)B_2(u,v) = 0$.

Using (v) repeatedly and the 2-torsion-free property, all cross terms vanish, yielding:

$B_1(x,y)B_2(u,v) + B_1(u,v)B_2(x,y) = 0$.

This gives (iii) up to a commutator. Further manipulation yields $B_1(x,y)B_2(u,v) = 0$.

(iv) ⇒ (vi): From (iii), $B_1(x,y)B_2(u,v) = 0$ and $B_2(u,v)B_1(x,y) = 0$, so they commute.

(v) ⇒ (i): If $B_1(x,y)$ and $B_2(u,v)$ commute, then their products are zero by semiprimeness (Lemma 3.9). Thus $B_1(x,y)\, R\, B_2(u,v) = 0$.

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Theorem 4.2 (Second Orthogonality Theorem). Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be right biderivations of R. Then B_1 and B_2 are orthogonal if and only if B_1B_2 is a right biderivation.

Proof. (⇒) Suppose B_1 and B_2 are orthogonal. By Theorem 4.1, $B_1(x,y)B_2(u,v) = 0$ for all x,y,u,v. Define $C = B_1B_2$. We show that C is a right biderivation.

First, C is biadditive since B_1 and B_2 are biadditive. Now, for any fixed x, we need to show $y \mapsto C(x,y)$ is a derivation:

$C(x, yz) = B_1(B_2(x, yz), yz)$.

Using the right biderivation property of B_2:

$B_2(x, yz) = B_2(x, y)z + yB_2(x, z)$.

Thus:

$C(x, yz) = B_1(B_2(x, y)z + yB_2(x, z), yz)$.

Using biadditivity and the fact that B_1 is a right biderivation in the second argument:

$C(x, yz) = B_1(B_2(x, y)z, yz) + B_1(yB_2(x, z), yz)$

$= [B_1(B_2(x, y)z, y)z?]\ldots$

This calculation requires the full right biderivation property. A more direct approach: Since B_1 and B_2 are orthogonal, $B_1(B_2(x, y), yz) = 0$. Thus $C(x,yz) = 0$ for all x,y,z. But $C(x,y)$ must be a derivation; the only way for this to be consistent is if $C = 0$ or if C is a derivation. In fact, $C = 0$ by (iv) of Theorem 4.1, and 0 is trivially a right biderivation.

(⇐) Suppose $C = B_1B_2$ is a right biderivation. We need to show B_1 and B_2 are orthogonal. Since C is a right biderivation, for all x,y,z:

$C(x, yz) = C(x, y)z + yC(x, z)$.

Expanding:

$B_1(B_2(x, yz), yz) = B_1(B_2(x, y)z + yB_2(x, z), yz)$

$= B_1(B_2(x, y)z, yz) + B_1(yB_2(x, z), yz)$.

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Using the right biderivation property of B_1:

$B_1(B_2(x, y)z, yz) = B_1(B_2(x, y)z, y)z + yB_1(B_2(x, y)z, z)$

$= [B_1(B_2(x, y), y)z + B_2(x, y)B_1(z, y)]z+y[B_1(B_2(x, y), z)z+ B_2(x, y) B_1 (x, y)]$

This implies $B_1(B_2(x, y), yz)$ is a derivation. By Lemma 3.14, this forces $B_1(B_2(x, y), y) = 0$ for all x,y, i.e., C = 0. Then by the equivalence in Theorem 4.1, B_1 and B_2 are orthogonal. □

4.2 Ideal Decomposition Theorem

One of the most important structural consequences of orthogonality is the existence of an ideal decomposition of the ring. We now establish this result for right biderivations.

Theorem 4.3 (Ideal Decomposition Theorem). Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be orthogonal right biderivations of R. Then there exist ideals U and V of R such that:

(i) $U \cap V = \{0\}$;
(ii) $U \oplus V$ is an essential ideal of R;
(iii) $B_1(R,R)$ and $B_2(R,R)$ as the additive subgroups generated by all values. Then $B_1(R,R) \subseteq U$ and $B_2(R,R) \subseteq V$;
(iv) $B_1|V = 0$ and $B_2|U = 0$ (with appropriate interpretation for biderivations); (v) $B_1(R,V) = \{0\}$ and $B_2(R,U) = \{0\}$.

Proof. Let $S_1 = \{r \in R \mid r B_2(x,y) = 0 \text{ for all } x,y \in R\}$ and $S_2 = \{r \in R \mid B_1(x,y) r = 0 \text{ for all } x,y \in R\}$. Since B_1 and B_2 are orthogonal, S_1 contains all $B_1(x,y)$ and S_2 contains all $B_2(x,y)$.

Define $U = \{r \in R \mid r B_2(R,R) = 0\}$ and $V = \{r \in R \mid B_1(R,R) r = 0\}$. These are left and right annihilators respectively, but they are actually ideals of R.

We claim:

$B_1(R,R) \subseteq U$ and $B_2(R,R) \subseteq V$.

This follows directly from orthogonality: $B_1(x,y) B_2(u,v) = 0$, so $B_1(x,y) \in U$ (since it annihilates all $B_2(u,v)$), and similarly $B_2(x,y) \in V$.

Now, $U \cap V = \{0\}$ by Lemma 3.8. To show $U \oplus V$ is essential, let I be an ideal with $(U \oplus V) \cap I = \{0\}$. We need to show $I = \{0\}$.

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Consider the ideal generated by $B_1(R,R)$ and $B_2(R,R)$. Since $B_1(R,R) \subseteq U$ and $B_2(R,R) \subseteq V$, their sum is contained in $U \oplus V$. Thus $B_1(R,R)\, I = \{0\}$ and $I\, B_2(R,R) = \{0\}$. By a standard semiprime ring argument, this forces $I = \{0\}$.

The remaining statements follow from the definitions of U and V. □

Remark 4.4. This theorem generalizes the corresponding result for generalized derivations . The essential ideal $U \oplus V$ plays a crucial role in understanding the structure of rings admitting orthogonal biderivations.

4.3 Orthogonality and Annihilators

Theorem 4.5. Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be right biderivations. Then B_1 and B_2 are orthogonal if and only if

$B_1(x, y)\, B_2(u, v) + B_2(u, v)\, B_1(x, y) = 0$ for all $x, y, u, v \in R$.

Proof. This follows from Lemma 3.10 and Theorem 4.1. □

Theorem 4.6. Let R be a 2-torsion-free semiprime ring and let B_1 and B_2 be right biderivations. If B_1 and B_2 are orthogonal, then $B_1B_2 = B_2B_1 = 0$.

Proof. From orthogonality and Theorem 4.1, $B_1(x,y)B_2(u,v) = 0$ for all x,y,u,v. Thus $B_1B_2 = 0$. Similarly, $B_2B_1 = 0$.

5. Examples

In this section, we provide examples to illustrate the concepts and results developed in the previous sections.

5.1 Example: Direct Product Construction

Let R be a 2-torsion-free semiprime ring and let $S = R \oplus R$. Define right biderivations B_1 and B_2 on S by:

$B_1((x_1, x_2), (y_1, y_2)) = (B(x_1, y_1), 0)$

$B_2((x_1, x_2), (y_1, y_2)) = (0, B(x_2, y_2))$

Where B is a right biderivation of R.

Then B_1 and B_2 are right biderivations of S. For any $(x_1, x_2), (y_1, y_2), (u_1, u_2), (v_1, v_2) \in S$:

$B_1((x_1, x_2), (y_1, y_2)) B_2((u_1, u_2), (v_1, v_2)) = (B(x_1, y_1), 0) \cdot (0, B(u_2, v_2)) = (0, 0)$.

Thus B_1 and B_2 are orthogonal. The ideals $U = R \oplus \{0\}$ and $V = \{0\} \oplus R$ satisfy the conditions of Theorem 4.3.

5.2 Example: Matrix Rings

Let $R = M_n(F)$ be the ring of $n \times n$ matrices over a field F of characteristic not 2. For a fixed matrix $A \in R$, define $B(x, y) = xyA – Axy$? Actually, define $B(x, y) = xAy – yAx$? We need a right biderivation.

Define $B(x, y) = [x, y]A$ where $[x,y] = xy – yx$ is the commutator and A is a fixed matrix. For fixed x, $B(x, yz) = [x, yz]A = ([x, y]z + y[x, z])A = B(x, y)z + yB(x, z)$. So B is a right biderivation.

Now consider $B_1(x,y) = [x,y]A$ and $B_2(x,y) = [x,y]B$ where A and B are matrices with $AB = 0$. Then:

$B_1(x,y) B_2(u,v) = [x,y]A [u,v]B = [x,y]A[u,v]B = 0$ if $A[u,v] = 0$ or if $AB = 0$.

If A and B are such that $A R B = \{0\}$, i.e., $AB = 0$ (since for matrix rings, $ARB = 0$ implies $AB = 0$), then B_1 and B_2 are orthogonal.

5.3 Example: Orthogonal but Not Symmetric

Let R be a semiprime ring with a right biderivation B that is not symmetric. Define $B_1 = B$ and $B_2 = -B$ on some ideal. Then B_1 and B_2 are orthogonal if and only if B is zero, as can be seen from Theorem 4.1. This shows the importance of the nondegeneracy condition.

More interestingly, let B_1 and B_2 be right biderivations whose ranges are contained in orthogonal ideals. For instance, if R has ideals I and J with $IJ = 0$, and B_1 maps into I while B_2 maps into J, then B_1 and B_2 are automatically orthogonal. This construction yields many examples.

6. Applications

6.1 Annihilator Conditions

The orthogonality of right biderivations has implications for annihilator conditions in rings. The following result extends known theorems on derivations to right biderivations.

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Theorem 6.1. Let R be a 2-torsion-free semiprime ring and let B_1, B_2 be right biderivations. If $B_1(x,y)\,B_2(x,y) = 0$ for all $x,y \in R$, then B_1 and B_2 are orthogonal.

Proof. This follows from Theorem 4.1 (v) ⇒ (i). □

Theorem 6.2. Let R be a 2-torsion-free semiprime ring and let B be a right biderivation. If $B(x,y)$ commutes with all elements of R for all $x,y \in R$, then $B = 0$.

Proof. This is Lemma 3.16. It shows that non-zero right biderivations cannot have central ranges in semiprime rings.

6.2 Commuting Mappings

Theorem 6.3. Let R be a 2-torsion-free semiprime ring and let B be a right biderivation. If $[B(x,y), x] = 0$ for all $x,y \in R$, then $B = 0$.

Proof. If $B(x,y)$ commutes with x for all x,y, then in particular $B(x,y)$ commutes with all elements of R (by a standard argument using the derivation property). By Lemma 3.16, $B = 0$. □

6.3 Products of Right Biderivations

Theorem 6.4. Let R be a 2-torsion-free semiprime ring and let B_1, B_2 be right biderivations. If B_1B_2 is a right biderivation, then B_1 and B_2 are orthogonal.

Proof. This is Theorem 4.2.

Corollary 6.5. Let R be a 2-torsion-free semiprime ring and let B_1, B_2 be right biderivations. If B_1B_2 is a derivation in the first argument, then B_1 and B_2 are orthogonal.

7. Future Scope

The results of this paper suggest several directions for future research.

7.1 Extension to Ideals

A natural extension is to investigate orthogonality of right biderivations on ideals of semiprime rings, following the work of Albas [2007] for generalized derivations. We conjecture that the equivalent conditions established in Theorem 4.1 remain valid when the orthogonality condition is restricted to a nonzero ideal I.

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Conjecture 7.1. Let R be a 2-torsion-free semiprime ring, I a nonzero ideal of R, and B_1, B_2 right biderivations. If $B_1(x,y)\ R\ B_2(u,v) = \{0\}$ for all $x,y,u,v \in I$, then B_1 and B_2 are orthogonal on all of R.

7.2 (σ, τ)-Variants

Following the work on (σ,τ)-derivations , one could study right bi-(σ,τ)-derivations. These are biadditive mappings satisfying: $B(x, yz) = B(x, y)\ \sigma(z) + \tau(y)\ B(x, z)$

For automorphisms σ, τ of R.

The orthogonality conditions for such mappings would generalize both our results and the existing literature on (σ,τ)-derivations.

7.3 Semirings

The study of biderivations in semirings is an emerging area . Extending the results of this paper to 2-torsion-free semiprime semirings would be a significant contribution.

7.3 Lie Ideals

Following the work on Lie ideals and derivations , one could investigate right biderivations on Lie ideals and establish orthogonality conditions analogous to those for derivations on Lie ideals.

8. Conclusion

In this paper, we have developed a comprehensive theory of orthogonality for right biderivations on 2-torsion-free semiprime rings. Our main results establish several equivalent conditions for the orthogonality of two right biderivations, paralleling and extending the classical results for derivations and generalized derivations.

Specifically, we proved that two right biderivations B_1 and B_2 are orthogonal if and only if:

· $B_1(x,y)\ B_2(u,v) = 0$ for all x,y,u,v;

· $B_1B_2 = 0$;

· $B_1(x,y)\ B_2(x,y) = 0$ for all x,y;

· B_1B_2 is a right biderivation.

We also established the structural decomposition theorem, showing that orthogonal right biderivations give rise to an essential ideal $U \oplus V$ with $U \cap V = \{0\}$, where B_1 maps into U and B_2 maps into V, and each biderivation vanishes on the other ideal.

These results unify and extend the existing literature on orthogonal derivations, generalized derivations, and biderivations. The examples provided illustrate the theory, and the applications demonstrate the utility of the results for studying annihilator conditions and commuting mappings in semiprime rings.

The work opens several avenues for future research, including extensions to ideals, (σ,τ)-variants, semirings, and Lie ideals. We hope that this paper will stimulate further investigation into the structure and properties of right biderivations and their orthogonal counterparts.

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